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The energy of unit vector fields on the 3-sphere

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Abstract

The stability of the three-dimensional Hopf vector field, as a harmonic section of the unit tangent bundle is viewed from a number of different angles. The spectrum of the vertical Jacobi operator is computed, and compared with that of the Jacobi operator of the identity map on the 3-sphere. The variational behaviour of the three-dimensional Hopf vector field is compared and contrasted with that of the closely related Hopf map. Finally, it is shown that the Hopf vector fields are the unique global minima of the energy functional restricted to unit vector fields on the 3-sphere.

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1. Introduction

A smooth unit vector field σ on a compact Riemannian manifold (M, g) with Euler characteristic zero may be regarded as a smooth mapping of Riemannian manifolds $\sigma : (M, g) \to (UM, h)$, where *UM* is the unit tangent bundle and h is the restriction of the Sasaki metric on the tangent bundle *TM*. The energy of σ may be defined accordingly. Since metrics h and g are horizontally isometric, and σ is a section, it suffices to consider the *vertical energy functional*:

$$E^{\mathrm{v}}(\sigma) = \int_M |\,\mathrm{d}^{\mathrm{v}}\sigma|^2\,\mathrm{d}x, \tag{1.1}$$

where $\mathrm{d}^{\mathrm{v}}\sigma$ is the vertical component of the differential $\mathrm{d}\sigma$. (Here, 'horizontal' and 'vertical' refer to the complementary distributions on *TM* defined by the Levi–Civita connection.) One says that σ is a critical point of E^{v}, or a *harmonic section* of *UM*, if E^{v} is stationary at

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σ with respect to variations through unit vector fields. The (non-linear) Euler–Lagrange equations for this variational problem are [17]

$$\nabla^*\nabla\sigma - |\nabla\sigma|^2\sigma = 0, \tag{1.2}$$

where $\nabla^*\nabla$ is the *trace* (or *rough*) *Laplacian*:

$$\nabla^*\nabla\sigma = -\mathrm{Tr}\nabla^2\sigma.$$

Further, one says that a harmonic section σ is E^{v}*-stable* if the second variation of E^{v} at σ with respect to unit vector fields is non-negative. The second variation of E^{v} in this constrained sense may be regarded as a quadratic form $\mathcal{H}^{\mathrm{v}}_\sigma$ (the *vertical Hessian*) on the space $\mathcal{V}_\sigma$ of appropriate variation fields; since σ is allowed to vary only through unit vector fields, $\mathcal{V}_\sigma$ is the space of smooth vector fields on M which are pointwise orthogonal to σ. Associated to $\mathcal{H}^{\mathrm{v}}_\sigma$ is the *vertical Jacobi operator* $\mathcal{J}^{\mathrm{v}}_\sigma$:

$$\mathcal{H}^{\mathrm{v}}_\sigma(\alpha,\beta) = \int_M \langle \mathcal{J}^{\mathrm{v}}_\sigma(\alpha),\beta\rangle\,\mathrm{d}x \quad \text{for all}\,\alpha,\beta\in\mathcal{V}_\sigma. \tag{1.3}$$

(Diamond brackets denote the relevant Riemannian metric, in this case g.) Explicit computation (see [17]) shows that $\mathcal{J}^{\mathrm{v}}_\sigma$ is the following symmetric, elliptic linear second-order partial differential operator on $\mathcal{V}_\sigma$:

$$\mathcal{J}^{\mathrm{v}}_\sigma(\alpha) = \nabla^*\nabla\alpha - |\nabla\sigma|^2\alpha - 2\langle\nabla\sigma,\nabla\alpha\rangle\sigma. \tag{1.4}$$

Thus, $\mathcal{J}^{\mathrm{v}}_\sigma$ may be viewed as a twisted version of the conventional Jacobi operator for a sphere-valued harmonic map [14]; indeed, since everything mentioned so far is in fact true for unit sections σ of any Riemannian vector bundle, this constitutes a complete generalization of the theory of harmonic maps into spheres. There is a unique extension of $\mathcal{J}^{\mathrm{v}}_\sigma$ to a self-adjoint linear operator on the Hilbert space of L^2 variation fields of σ.

The *canonical Hopf vector field* on $M = S^{2n+1}$ ($n = 0, 1, 2, \dots$) is defined:

$$\sigma(x) = \mathrm{i}x, \quad x\in\mathbb{R}^{2n+2}\cong\mathbb{C}^{n+1},\ |x| = 1, \tag{1.5}$$

where $\mathrm{i} = \sqrt{-1}$. More generally, any unit vector field congruent to σ will be called a *Hopf vector field*. It is known that in all dimensions σ is a harmonic section, and if $n = 2, 3, \dots$, then σ is E^{v}-unstable (see [17]). This comes as no surprise, in view of Xin's [19] instability theorem for harmonic mappings from spheres. More surprising is the fact that the three-dimensional Hopf vector field is E^{v}-stable [18]. In this paper, we take a closer look at $\mathcal{J}^{\mathrm{v}}_\sigma$ when σ is the three-dimensional Hopf vector field, confirming its non-negativity in a number of different ways. In Section 2 we make the interesting observation that the non-negativity of $\mathcal{J}^{\mathrm{v}}_\sigma$ is purely a consequence of dimension and curvature, and the fact that σ has geodesic integral curves; the Lie group structure of S^3 is in fact only incidental (although it was used in [18]). Our primary aim in Section 2, however, is to compute the (necessarily discrete) spectrum of $\mathcal{J}^{\mathrm{v}}_\sigma$, along with the eigenvalue multiplicities (Theorem 2.3). It is interesting to compare these spectral data with those of the Jacobi operator of the identity map id on S^3; for, the Hessian $\mathcal{H}$ of the latter is related to $\mathcal{H}^{\mathrm{v}}_\sigma$ by the restriction:

$$\mathcal{H}^{\mathrm{v}}_\sigma = \mathcal{H}(\mathcal{V}_\sigma,\mathcal{V}_\sigma).$$

Recall that the space of variation fields for id is the entire Lie algebra of vector fields on S^3; of course, since id is E-unstable, $\mathcal{H}$ is indefinite. Thus, we have a rather nice example illustrating how a 'simple' change of domain can dramatically alter the spectrum (see also Remark 2.4). In addition to this close relationship between the variational aspects of σ and id, there is also a strong linkage between σ and another E-unstable harmonic map: the Hopf map $\varphi : S^3 \to S^2$. At first sight (see Proposition 3.1) this link seems so compelling that one is tempted to infer that the variational properties of σ must surely be the same as those of φ. In Section 3, we tease out the subtleties of the relationship, and show how σ manages to avoid the instabilities of φ. In addition, we compute the spectrum of the Jacobi operator of φ, correcting errors of [15] (see Remark 3.12). Finally, in Section 4, we cap-off the discussion of $\mathcal{J}_\sigma^{\mathrm{v}}$ with the following much more powerful global result.

Main theorem. *The absolute minimum of E^{v} over all unit vector fields on S^3* is $2\pi^2$, *which is achieved at, and only at, the Hopf vector fields*.

This theorem provides an affirmative resolution of a conjecture in [18], and establishes a complete analogy between the behaviour of the energy and volume functionals on the space of unit vector fields on S^3 [9]. Our proof of the Main theorem utilizes a Rigidity theorem for shear-free geodesic congruences on S^3, which follows from the work of Baird and Wood (combining results of [2,3]) on harmonic morphisms. However, we take the opportunity to provide a direct proof of this Rigidity theorem.

2. The spectrum of the vertical Jacobi operator

We refer to the following well-known diffeomorphism of S^3 with $SU(2)$ as the *Pauli correspondence*:

$$(z_1, z_2) \leftrightarrow \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}, \quad z_1, z_2 \in \mathbb{C},\ |z_1|^2 + |z_2|^2 = 1,$$

and the following basis of the Lie algebra $\mathfrak{su}(2)$ as the *Pauli basis*:

$$P_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Pauli basis satisfies the commutation relations:

$$[P_j, P_k] = 2\epsilon_{jkl} P_l, \tag{2.1}$$

where ϵ_{jkl} is the totally antisymmetric symbol. There exists a unique bi-invariant metric on $SU(2)$ rendering the Pauli basis orthonormal, and with respect to this metric the Pauli correspondence is an isometry. Furthermore, with respect to the orientation of $SU(2)$ determined by the Pauli basis, the Pauli correspondence preserves the orientation of S^3 induced from the standard orientation of $\mathbb{R}^4$ by the outward-pointing normal. The Hopf vector fields on S^3 are precisely the Pauli-preimages of unit vector fields on $SU(2)$ which are either left-invariant or right-invariant.

Let σ_j denote the Pauli-preimage of the left-invariant vector field on $SU(2)$ generated by P_j. Then $(\sigma_1, \sigma_2, \sigma_3)$ is a positively oriented global orthonormal frame on S^3, with $\sigma_1 = \sigma$, the canonical Hopf vector field. It follows from left-invariance (see Lemma 3.2) that

$$\nabla_{\sigma_j}\sigma_k = \tfrac{1}{2}[\sigma_j, \sigma_k] = \epsilon_{jkl}\sigma_l \quad \text{by (2.1)}. \tag{2.2}$$

It follows from (2.2) that

$$|\nabla\sigma|^2 = 2. \tag{2.3}$$

In order to compute the spectrum of $\mathcal{J}_\sigma^{\mathrm{v}}$, we first rewrite this operator using the triad basis $\{\sigma_j\}$, noting that any $\alpha \in \mathcal{V}_\sigma$ can be written as $\alpha = f_2\sigma_2 + f_3\sigma_3$, where f_2 and f_3 are smooth real-valued functions on S^3. The following fact will be useful.

Lemma 2.1. *If X (resp. λ) is any smooth vector field (resp. function) on a Riemannian manifold, then*

$$\nabla^*\nabla(\lambda X) = (\Delta\lambda)X + \lambda\nabla^*\nabla X - 2\nabla_{\mathrm{grad}\,\lambda}X,$$

where Δ is the Laplace–Beltrami operator acting on functions.

Proposition 2.2. *Let*

$$\alpha = f_2\sigma_2 + f_3\sigma_3, \tag{2.4}$$

where f_2 and f_3 are smooth real functions on S^3. Then

$$\mathcal{J}_\sigma^{\mathrm{v}}(\alpha) = (\Delta f_2 + 2\nabla_\sigma f_3)\sigma_2 + (\Delta f_3 - 2\nabla_\sigma f_2)\sigma_3.$$

Proof. It follows from (1.4) and (2.3) that

$$\mathcal{J}_\sigma^{\mathrm{v}}(\alpha) = \nabla^*\nabla\alpha - 2\alpha - 2\langle\nabla\sigma, \nabla\alpha\rangle\sigma. \tag{2.5}$$

We first compute $(\nabla^*\nabla - 2)\alpha$. By Lemma 2.1

$$\nabla^*\nabla\alpha = (\Delta f_2)\sigma_2 + (\Delta f_3)\sigma_3 + f_2\nabla^*\nabla\sigma_2 + f_3\nabla^*\nabla\sigma_3 - 2(\nabla_{\mathrm{grad}\,f_2}\sigma_2 + \nabla_{\mathrm{grad}\,f_3}\sigma_3).$$

Since σ_j is a Killing field, we have

$$\nabla^*\nabla\sigma_j = \mathrm{Ric}(\sigma_j) = 2\sigma_j.$$

(This can also be deduced from Eq. (2.2). Furthermore, by (2.2):

$$\nabla_{\mathrm{grad}\,f_2}\sigma_2 = \sum_{j=1}^{3}(\nabla_{\sigma_j} f_2)\nabla_{\sigma_j}\sigma_2 = \sum_{j,l}\epsilon_{j2l}(\nabla_{\sigma_j} f_2)\sigma_l = (\nabla_{\sigma_1} f_2)\sigma_3 - (\nabla_{\sigma_3} f_2)\sigma_1,$$

$$\nabla_{\mathrm{grad}\,f_3}\sigma_3 = \sum_{j,l}\epsilon_{j3l}(\nabla_{\sigma_j} f_3)\sigma_l = (\nabla_{\sigma_2} f_3)\sigma_1 - (\nabla_{\sigma_1} f_3)\sigma_2.$$

Therefore

$$\begin{aligned}(\nabla^*\nabla - 2)\alpha &= (\Delta f_2)\sigma_2 + (\Delta f_3)\sigma_3 - 2((\nabla_\sigma f_2)\sigma_3 \\ &\quad -(\nabla_\sigma f_3)\sigma_2) + 2(\nabla_{\sigma_3} f_2 - \nabla_{\sigma_2} f_3)\sigma.\end{aligned} \tag{2.6}$$

Next, we compute $\langle\nabla\sigma, \nabla\alpha\rangle$. By (2.2) again

$$\langle\nabla\sigma, \nabla\alpha\rangle = \sum_{j=1}^{3}\langle\nabla_{\sigma_j}\sigma, \nabla_{\sigma_j}\alpha\rangle = -\langle\sigma_3, \nabla_{\sigma_2}\alpha\rangle + \langle\sigma_2, \nabla_{\sigma_3}\alpha\rangle.$$

Hence

$$\langle\nabla\sigma, \nabla\alpha\rangle = \nabla_{\sigma_3} f_2 - \nabla_{\sigma_2} f_3.$$

Therefore the term proportional to σ in (2.6) is cancelled by $-2\langle\nabla\sigma, \nabla\alpha\rangle\sigma$ in (2.5) as it should. Hence, the result follows. □

Now, let us define a map ψ from the space of smooth vector fields pointwise orthogonal to σ to the space of smooth complex functions on S^3 by

$$\psi : \alpha \to f = f_2 + \mathrm{i} f_3.$$

The space of $\mathbb{C}$-valued functions on S^3 is regarded as a real Hilbert space, using the inner product

$$(f, g) = \mathrm{Re}\int_{S^3} f\bar{g}\,\mathrm{d}x,$$

where $\bar{f}$ is the complex conjugate of f. Then the proposition implies that

$$\mathcal{J}_\sigma^{\mathrm{v}} = \psi^{-1}\circ\Lambda\circ\psi,$$

where the map Λ on the space of complex functions is defined by

$$\Lambda(f) = \Delta f - 2\mathrm{i}\nabla_\sigma f.$$

Hence, the spectrum of $\mathcal{J}_\sigma^{\mathrm{v}}$ is identical to that of Λ. Since the operators Δ and ∇_σ commute, the complete set of eigenfunctions of Λ can be chosen to be simultaneous eigenfunctions of Δ and ∇_σ. The eigenvalues of Δ are $n(n+2)$, $n = 0, 1, 2, \ldots$, and eigenfunctions corresponding to each n form the representation $(\frac{1}{2}n, \frac{1}{2}n)$ of $SU(2)\otimes SU(2)$, with multiplicity $2(n+1)^2$. (Here, since the functions f and $\mathrm{i}f$ are linearly independent, the multiplicity is doubled compared to the usual case.) It is known that the eigenvalues of $\mathrm{i}\nabla_\sigma$ on scalar functions are $-n, -n+2, \ldots, n-2, n$, each with multiplicity $2(n+1)$. Thus, we have established the following theorem.

Theorem 2.3. *The eigenvalues of $\mathcal{J}_\sigma^{\mathrm{v}}$, where σ is a Hopf vector field, are $n(n+2)+2k$, where n is a non-negative integer and $k = -n, -n+2, \ldots, n-2, n$. The multiplicity of each eigenvalue for given n and k is $2(n+1)$.*

We note that two eigenvalues with different n and k are distinct. In fact, the eigenvalue sequence is a splice of the two arithmetic progressions $4k, 4k+1, k = 0, 1, 2, \ldots$. Since the eigenvalues are all non-negative, it follows that $\mathcal{J}^{\mathrm{v}}_{\sigma}$ is non-negative. The zero eigenvalue corresponds to $f = \text{const}$. Thus, $\mathcal{J}^{\mathrm{v}}_{\sigma}(\alpha) = 0$ if and only if α is a linear combination of σ_2 and σ_3; in particular, α is left-invariant.

Remark 2.4. *The Jacobi operator $\mathcal{J}$ for the identity map on S^3 is* [14]:

$$\mathcal{J} = \nabla^*\nabla - 2 = \Delta - 4$$

by application of a Weitzenböck formula, where Δ is the Hodge–de Rham Laplacian acting on the entire space of 1*-forms/vector fields. The eigenvalues of Δ fall into two sequences*:

$$\lambda_k = (k+1)(k+3), \qquad \tilde{\lambda}_k = (k+2)^2, \quad k = 0, 1, 2, \ldots$$

with corresponding multiplicities:

$$\mu_k = (k+2)^2, \qquad \tilde{\mu}_k = 2(k+1)(k+3), \quad k = 0, 1, 2, \ldots .$$

(*The eigenvectors of λ_k are exact* 1*-forms, whereas those of $\tilde{\lambda}_k$ are co-exact; see* [4] *with corrections of* [7], *or* [12]). *Therefore the eigenvalues of $\mathcal{J}$ are*

$$k^2 + 4k - 1, \qquad k^2 + 4k, \quad k = 0, 1, 2, \ldots$$

with multiplicities $\mu_k, \tilde{\mu}_k$, respectively. Notice that these two sequences are qualitatively quite different from those of Theorem 2.3.

The non-negativity of $\mathcal{J}^{\mathrm{v}}_{\sigma}$ was established in a slightly different manner in [18] using the Bochner–Yano integral formula [5] on any compact manifold (M, g):

$$\int_M (|\nabla X|^2 - \mathrm{Ric}(X, X))\,\mathrm{d}x = \int_M \left(\tfrac{1}{2}|L_X g|^2 - (\mathrm{div}\,X)^2\right)\mathrm{d}x. \tag{2.7}$$

This was used to rewrite (1.3) on S^3 as

$$\mathcal{H}^{\mathrm{v}}_{\sigma}(\alpha, \alpha) = \int_{S^3} \left(\tfrac{1}{2}|L_\alpha g|^2 - (\mathrm{div}\,\alpha)^2\right)\mathrm{d}x \tag{2.8}$$

if σ is a Hopf vector field. Then, the expansion (2.4) was used to show the non-negativity of $\mathcal{H}^{\mathrm{v}}_{\sigma}$, and to derive the condition on α for $\mathcal{H}^{\mathrm{v}}_{\sigma}(\alpha, \alpha) = 0$. In fact, the non-negativity of the *integrand* of (2.8) holds with a weaker condition on σ. We conclude this section by establishing this fact. (A similar inequality will be used in Section 4 to prove the Main theorem, mentioned in Section 1.)

Proposition 2.5 (Linear inequality). *Let σ be a unit vector field on an n-dimensional Riemannian manifold (M, g). If the integral curves of σ are geodesics, then for any vector field α pointwise orthogonal to σ we have*

$$(n-1)|L_\alpha g|^2 \geq 4(\mathrm{div}\,\alpha)^2.$$

Proof. Recall first that

$$L_X g(Y, Z) = \langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle \tag{2.9}$$

for any vector field X. It follows that

$$\mathrm{Tr}\, L_X g = 2\mathrm{div}\, X. \tag{2.10}$$

Let $\alpha_1, \dots, \alpha_{n-1}$ be local vector fields such that $(\alpha_1, \dots, \alpha_{n-1}, \sigma)$ is an orthonormal moving frame on M. It follows from (2.10) that

$$\begin{aligned}(n-1)|L_X g|^2 - 4(\mathrm{div}\, X)^2 &= (n-1)|L_X g|^2 - (\mathrm{Tr}\, L_X g)^2 \\ &= (n-2)(L_X g(\sigma,\sigma)^2 + \sum_i L_X g(\alpha_i,\alpha_i)^2) \\ &\quad +2(n-1)\sum_i L_X g(\sigma,\alpha_i)^2 + (n-1)\sum_{i\neq j} L_X g(\alpha_i,\alpha_j)^2 \\ &\quad -2\sum_i L_X g(\sigma,\sigma) L_X g(\alpha_i,\alpha_i) \\ &\quad -2\sum_{i<j} L_X g(\alpha_i,\alpha_i) L_X g(\alpha_j,\alpha_j).\end{aligned} \tag{2.11}$$

Since the integral curves of σ are geodesics, it follows from (2.9) that

$$L_\alpha g(\sigma, \sigma) = 2\langle \nabla_\sigma \alpha, \sigma \rangle = -2\langle \alpha, \nabla_\sigma \sigma \rangle = 0. \tag{2.12}$$

Therefore if $X = \alpha$ then (2.11) collapses to

$$\begin{aligned}(n-1)|L_X g|^2 - 4(\mathrm{div}\, X)^2 &= 2(n-1) L_X g(\sigma,\alpha_i)^2 + (n-1)\sum_{i\neq j} L_X g(\alpha_i,\alpha_j)^2 \\ &\quad +\sum_{i<j}(L_X g(\alpha_i,\alpha_i) - L_X g(\alpha_j,\alpha_j))^2 \geq 0.\end{aligned} \tag{2.13}$$

□

Note that the non-negativity of the integrand in (2.8) follows from the case $n = 3$ of this proposition. Although there are many unit vector fields on S^3 whose integral curves are geodesics [8], apart from the Hopf vector fields we do not know how many of them are harmonic sections of US^3. In fact, this is a very interesting open question.

3. Comparison with the Hopf map

We first note an extremely simple relationship between the three-dimensional canonical Hopf vector field σ and the Hopf map $\varphi : S^3 \to S^2$. Recall that on any Lie group G there is the Maurer–Cartan form μ, with values in the Lie algebra:

$$\mu(X) = g^{-1} \cdot X \quad \forall X \in T_g G,\ \forall g \in G.$$

Let us denote by η the right-invariant analogue

$$\eta(X) = X \cdot g^{-1}.$$

Throughout Section 3 we identify S^3 with $SU(2)$ via the Pauli correspondence, as described at the beginning of Section 2. Then σ is left-invariant; therefore $\mu \circ \sigma = \text{const.}$ (the Pauli matrix P_1). On the other hand, we have the following proposition.

Proposition 3.1. *There exists an isometric identification of S^2 with the unit sphere in $\mathfrak{su}(2)$ such that $\eta \circ \sigma = \varphi$, the Hopf map $S^3 \to S^2$.*

Proof. If the Pauli-image of $(z_1, z_2) \in S^3$ is denoted by γ, and e denotes the identity, then

$$\begin{aligned}\eta \circ \sigma(\gamma) = \gamma \cdot \sigma(e) \cdot \gamma^{-1} &= \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \overline{z_1} & \overline{z_2} \\ -z_2 & z_1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{i}|z_1|^2 - \mathrm{i}|z_2|^2 & 2\mathrm{i}z_1\overline{z_2} \\ 2\mathrm{i}\overline{z_1}z_2 & \mathrm{i}|z_2|^2 - \mathrm{i}|z_1|^2 \end{pmatrix} \\ &= (|z_1|^2 - |z_2|^2)P_1 + 2\,\mathrm{Re}(\overline{z_1}z_2)P_2 + 2\,\mathrm{Im}(\overline{z_1}z_2)P_3. \end{aligned} \quad (3.1)$$

On the other hand, by the Hopf map we understand the composition of the Hopf fibration $\pi : S^3 \to \mathbb{C}P^1; (z_1, z_2) \mapsto (z_1 : z_2)$ (homogeneous coordinates) with the standard isomorphism of $\mathbb{C}P^1$ with the unit sphere in $\mathbb{R}^3$. This isomorphism admits the following local description: on the open subset of $\mathbb{C}P^1$ where $z_1 \neq 0$, the complex chart $(z_1 : z_2) \mapsto z_2/z_1$ is followed by the inverse of stereographic projection from the north pole onto the equatorial plane. Thus

$$\varphi(z_1, z_2) = (2\overline{z_1}z_2, |z_2|^2 - |z_1|^2) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3.$$

But by (3.1) this is precisely the map $S^3 \to S^2$ obtained by taking the coordinates of $\eta \circ \sigma$ with respect to the basis $(P_2, P_3, -P_1)$ of $\mathfrak{su}(2)$. □

Note. The most natural identification of the unit sphere in $\mathfrak{su}(2)$ with S^2 (viz. the restriction of the linear isomorphism which sends the Pauli basis (P_1, P_2, P_3) to the standard basis of $\mathbb{R}^3$) is not the identification asserted by Proposition 3.1; rather, the two differ by a rotary reflection of S^2.

At first glance, the fact that σ differs from φ 'only' by right translation suggests a complete equivalence between the variational theories of $E^{\mathrm{v}}(\sigma)$ and $E(\varphi)$. Of course, in the light of Section 2 we know this is not the case, since φ is E-unstable. A closer examination of the energy functionals (see Proposition 3.4), and Jacobi operators (see Proposition 3.7) reveals a more complicated scenario, which resolves the issue. Recall first the following characterizations of covariant differentiation in Lie groups [10].

Lemma 3.2. *For any Lie group with bi-invariant metric, the Levi–Civita connection may be characterized in either of the following two ways*:

1. $\mu(\nabla_X Y) = \mathrm{d}(\mu Y)(X) + \frac{1}{2}[\mu X, \mu Y]$.
2. $\eta(\nabla_X Y) = \mathrm{d}(\eta Y)(X) - \frac{1}{2}[\eta X, \eta Y]$.

Lemma 3.3. *If Y is a vector field on any Lie group with bi-invariant metric, then*

$$|\mathrm{d}(\eta Y)(X)|^2 = 2|\nabla_X Y|^2 - |\,\mathrm{d}(\mu Y)(X)|^2 + \tfrac{1}{2}|[\mu X, \mu Y]|^2.$$

In particular, on $S^3 \cong SU(2)$ we have

$$|\mathrm{d}(\eta Y)|^2 = 4|Y|^2 + 2|\nabla Y|^2 - |\mathrm{d}(\mu Y)|^2.$$

Proof. It follows from Lemma 3.2(2) and the bi-invariance of the metric that

$$\begin{aligned}|\mathrm{d}(\eta Y)(X)|^2 &= |\nabla_X Y|^2 + \langle [\eta X, \eta Y], \eta(\nabla_X Y)\rangle + \tfrac{1}{4}|[\eta X, \eta Y]|^2\\ &= |\nabla_X Y|^2 + \langle [\mu X, \mu Y], \mu(\nabla_X Y)\rangle + \tfrac{1}{4}|[\mu X, \mu Y]|^2.\end{aligned}$$

Now use the algebraic fact that if a, b, c are vectors in any inner product space, with $a = b + c$, then

$$2\langle a, c\rangle = |a|^2 - |b|^2 + |c|^2.$$

Lemma 3.2(1) permits the choice

$$a = \mu(\nabla_X Y), \qquad b = \mathrm{d}(\mu Y)(X), \qquad c = \tfrac{1}{2}[\mu X, \mu Y]$$

and the general identity follows. This identity implies

$$|\mathrm{d}(\eta Y)|^2 = 2|\nabla Y|^2 - |\mathrm{d}(\mu Y)|^2 + \tfrac{1}{2}\sum_j |[\mu E_j, \mu Y]|^2,$$

where $\{E_j\}$ is any orthonormal tangent frame. On $SU(2)$, let $\{E_j\}$ be the global left-invariant orthonormal frame $\{\sigma_j\}$, and write $Y = Y^k\sigma_k$ (summation convention). By the commutation relations (2.1):

$$\sum_j |[\mu E_j, \mu Y]|^2 = \sum_{j,k} |[P_j,\ Y^k P_k]|^2 = \sum_{j,k,l} 4|\epsilon_{jkl} Y^k P_l|^2 = 8\sum_k (Y^k)^2 = 8|Y|^2. \quad \square$$

Remark. *It follows from Proposition* 3.1 *and Lemma* 3.3 (*taking $Y = \sigma$*) *that*

$$|\mathrm{d}\varphi|^2 = 4 + 2|\nabla\sigma|^2 = 8$$

and we recover the familiar fact that the Hopf map $\varphi : S^3 \to S^2$ is an 'eigenmap' with eigenvalue 8 [6].

Proposition 3.4. *Let σ_t be any variation of the Hopf vector field on S^3 through unit vector fields, and let $\varphi_t = \eta \circ \sigma_t$ be the corresponding variation of the Hopf map* (*cf. Proposition* 3.1). *Then*

$$E(\varphi_t) = 4\pi^2 + 2E^{\mathrm{v}}(\sigma_t) - E(\mu \circ \sigma_t).$$

Proof. Integrate Lemma 3.3 with $Y = \sigma_t$, recalling that $|\mathrm{d}^{\mathrm{v}}\sigma_t|^2 = |\nabla\sigma_t|^2$ (see [17]), and S^3 has volume $2\pi^2$. $\square$

It follows from Proposition 3.4 that if σ_t is E^{v}-decreasing, then φ_t is E-decreasing. However, the converse is not necessarily true. Indeed, since Proposition 3.4 tells us that E^{v} is essentially the average of the energies of the right- and left-translates, it is conceivable that any E-decreasing variation of the right-translate φ of σ is compensated by an E-increasing variation of the left-translate. Our aim is to show that this is indeed the case.

Before proceeding further, we make some simple observations. Let $\langle\sigma\rangle$ denote the line subbundle of TS^3 generated by σ. Then $\langle\sigma\rangle = \ker \mathrm{d}\varphi$. Therefore $\mathcal{V}_\sigma$ is precisely the space of smooth *φ-horizontal* vector fields on S^3. This terminology will be very useful. For example, the covariant derivative of σ may be written as follows:

$$\nabla_X\sigma = \begin{cases} 0 & \text{if } X \text{ is } \varphi\text{-vertical}, \\ \mathrm{i}X & \text{if } X \text{ is } \varphi\text{-horizontal}. \end{cases} \tag{3.2}$$

The space of L^2 variation fields for φ admits the following L^2-orthogonal decomposition

$$\mathcal{N}_\varphi \oplus \mathcal{Z}_\varphi \oplus \mathcal{P}_\varphi,$$

where $\mathcal{N}_\varphi$ (resp. $\mathcal{P}_\varphi$) is the direct sum of the negative (resp. positive) eigenspaces of the Jacobi operator $\mathcal{J}_\varphi$ of φ (see [14]), and $\mathcal{Z}_\varphi$ is the kernel of $\mathcal{J}_\varphi$. General elliptic theory guarantees that $\mathcal{N}_\varphi$ and $\mathcal{Z}_\varphi$ are finite-dimensional, and that all their elements are smooth sections (of the pullback bundle φ^*TS^2). The dimensions of $\mathcal{N}_\varphi$ and $\mathcal{Z}_\varphi$ were computed in [15] (modulo a few errors; see Remark 3.12), from which the following facts may be deduced.

(A) $\mathcal{N}_\varphi$ is four-dimensional; it comprises variation fields of the form $\mathrm{d}\varphi(\Gamma)$, where Γ is a conformal gradient field on S^3. (By a conformal gradient field on a sphere S^n we mean the spherical gradient of the restriction of a linear functional on the ambient Euclidean space $\mathbb{R}^{n+1}$.) Let $\mathcal{G}_\sigma$ denote the subspace of $\mathcal{V}_\sigma$ comprising the φ-horizontal components of conformal gradient fields on S^3 (no conformal gradient field is φ-horizontal). Then $\mathcal{N}_\varphi = \mathrm{d}\varphi(\mathcal{G}_\sigma)$.

(B) $\mathcal{Z}_\varphi$ is eight-dimensional; it is generated by variation fields of the form $\mathrm{d}\varphi(Z)$, where Z is an infinitesimal isometry of S^3, and of the form $C(\varphi)$ where C is a conformal vector field on S^2. The Lie algebra $\mathfrak{I}$ of infinitesimal isometries of S^3 is six-dimensional; it is generated by the vector fields on $SU(2)$ which are either left- or right-invariant. However, the fibres of φ are invariant under the flow of the left-invariant vector field σ, so $\mathrm{d}\varphi(\mathfrak{I})$ is actually five-dimensional. The Lie algebra $\mathfrak{C}$ of conformal vector fields on S^2 is also six-dimensional. We may write $C(\varphi) = \mathrm{d}\varphi(\tilde{C})$ where $\tilde{C}$ is the φ-horizontal lift of C. If C is an infinitesimal isometry of S^2 then $\tilde{C}$ is an infinitesimal isometry of S^3 (in fact, a right-invariant vector field), so in this case $C(\varphi) \in \mathrm{d}\varphi(\mathfrak{I})$. However, if C is a conformal gradient field of S^2 then the variation fields $C(\varphi)$ constitute a three-dimensional subspace of $\mathfrak{C}(\varphi)$ which is complementary to $\mathrm{d}\varphi(\mathfrak{I})$ in $\mathcal{Z}_\varphi$. (Note that $\tilde{C}$ is not a conformal field on S^3, unless C is an infinitesimal isometry.) If $\tilde{\mathfrak{C}}$ denotes the six-dimensional subspace of $\mathcal{V}_\sigma$ comprising the φ-horizontal lifts of elements of $\mathfrak{C}$, and an eight-dimensional subspace $\mathcal{F}_\sigma$ of $\mathcal{V}_\sigma$ is defined by

$$\mathcal{F}_\sigma = \tilde{\mathfrak{C}} \oplus \mathbb{R}\sigma_2 \oplus \mathbb{R}\sigma_3, \tag{3.3}$$

then it follows that $\mathcal{Z}_\varphi = \mathrm{d}\varphi(\mathcal{F}_\sigma)$.

The space of L^2 variation fields for σ also decomposes as an L^2-orthogonal direct sum:

$$\mathcal{N}_\sigma \oplus \mathcal{Z}_\sigma \oplus \mathcal{P}_\sigma,$$

where $\mathcal{N}_\sigma$ (resp. $\mathcal{P}_\sigma$) is the direct sum of the negative (resp. positive) eigenspaces of $\mathcal{J}_\sigma^{\mathrm{v}}$, and $\mathcal{Z}_\sigma$ is the kernel of $\mathcal{J}_\sigma^{\mathrm{v}}$. We would like to see how the subspaces $\mathcal{F}_\sigma$ and $\mathcal{G}_\sigma$ corresponding to low spectral frequencies of $\mathcal{J}_\varphi$ relate to the eigenspaces of $\mathcal{J}_\sigma^{\mathrm{v}}$ (see Proposition 3.8); when used in conjunction with our energy formula (Proposition 3.4), this will enable us to deduce that $\mathcal{N}_\sigma$ is trivial. We will derive this relationship by comparing the Jacobi operators $\mathcal{J}_\varphi$ and $\mathcal{J}_\sigma^{\mathrm{v}}$ (see Proposition 3.7).

Recall that the Jacobi operator $\mathcal{J}_f$ for an arbitrary harmonic mapping f of Riemannian manifolds is given by [14]:

$$\mathcal{J}_f(w) = \nabla^*\nabla w - \mathrm{Ric}_f(w),$$

where w is any variation field for f, and

$$\mathrm{Ric}_f(w) = \mathrm{Tr}\, R(w, \mathrm{d}f)\, \mathrm{d}f.$$

If the domain of f is a sphere S^{m+1}, and the variation field is of the form $w = \mathrm{d}f(X)$ where X is a vector field on S^{m+1}, then application of a Weitzenböck formula yields [19]:

$$\mathcal{J}_f(w) = \mathrm{d}f(\nabla^*\nabla X - mX) - 2\sum_j \nabla\, \mathrm{d}f(E_j, \nabla_{E_j} X), \tag{3.4}$$

where $\{E_j\}$ is any local orthonormal tangent frame in S^{m+1}. The term $\nabla\, \mathrm{d}f$ is sometimes referred to as the *second fundamental form* of f; when $f = \varphi$ it can be computed using standard results on Riemannian submersions.

Lemma 3.5. *Let* α, β (*resp.* V, W) *be* φ*-horizontal* (*resp.* φ*-vertical*) *tangent vectors of* S^3. *Then*
1. $\nabla \mathrm{d}\varphi(\alpha, \beta) = 0$,
2. $\nabla \mathrm{d}\varphi(V, W) = 0$,
3. $\langle \nabla \mathrm{d}\varphi(\alpha, V), \mathrm{d}\varphi(\beta)\rangle = 2\langle V, [\alpha, \beta]\rangle$.

Proof. The map $\frac{1}{2}\varphi$ is a Riemannian submersion. (By $k\varphi$ for any $k \in \mathbb{R}^+$ we mean the mapping onto the 2-sphere of radius k obtained by scalar multiplication in ambient $\mathbb{R}^3$.) Then (1) is an identity for all Riemannian submersions, (2) follows from the fact that φ has totally geodesic fibres, and (3) is a rescaling of the identity

$$\langle \nabla \mathrm{d}\pi(\alpha, V), \mathrm{d}\pi(\beta)\rangle = \tfrac{1}{2}\langle V, [\alpha, \beta]\rangle$$

for any Riemannian submersion π. (Standard results on the second fundamental form of a Riemannian submersion may be found in [1, Chapter 9], [11,13,16].) □

Let J denote the standard complex structure on the unit sphere S^2, characterized by the condition that φ is 'horizontally homothetically holomorphic':

$$\mathrm{d}\varphi(\mathrm{i}\alpha) = 2J\,\mathrm{d}\varphi(\alpha), \tag{3.5}$$

for all φ-horizontal vectors α.

Lemma 3.6. *For all vector fields X,Y on S^3, we have*

$$\nabla \mathrm{d}\varphi(X, Y) = -2J\,\mathrm{d}\varphi(\langle\sigma, X\rangle Y + \langle\sigma, Y\rangle X).$$

Proof. Note first that if α, β are φ-horizontal then:

$$\langle\sigma, [\alpha, \beta]\rangle = \langle\sigma, \nabla_\alpha\beta - \nabla_\beta\alpha\rangle = -\langle\nabla_\alpha\sigma, \beta\rangle + \langle\nabla_\beta\sigma, \alpha\rangle = -2\langle \mathrm{i}\alpha, \beta\rangle \quad \text{by (3.2)}.$$

If $\{\beta_j : j = 1, 2\}$ is any φ-horizontal orthonormal frame, then $\{\frac{1}{2}\mathrm{d}\varphi(\beta_j)\}$ is an orthonormal frame of S^2, and it follows from Lemma 3.5 and (3.5) that

$$\begin{aligned}\nabla \mathrm{d}\varphi(\sigma, \alpha) &= \tfrac{1}{4}\langle\nabla \mathrm{d}\varphi(\sigma, \alpha), \mathrm{d}\varphi(\beta_j)\rangle\, \mathrm{d}\varphi(\beta_j) = \tfrac{1}{2}\langle\sigma, [\alpha, \beta_j]\rangle\, \mathrm{d}\varphi(\beta_j)\\ &= -\langle \mathrm{i}\alpha, \beta_j\rangle\, \mathrm{d}\varphi(\beta_j) = -\mathrm{d}\varphi(\mathrm{i}\alpha) = -2J\,\mathrm{d}\varphi(\alpha).\end{aligned} \tag{3.6}$$

Now suppose that α, β are the φ-horizontal components of X, Y, respectively:

$$X = \alpha + \langle X, \sigma\rangle\sigma, \qquad Y = \beta + \langle Y, \sigma\rangle\sigma.$$

It follows from Lemma 3.5 that:

$$\begin{aligned}\nabla \mathrm{d}\varphi(X, Y) &= \langle X, \sigma\rangle\nabla \mathrm{d}\varphi(\sigma, \beta) + \langle Y, \sigma\rangle\nabla \mathrm{d}\varphi(\alpha, \sigma)\\ &= -2\langle\sigma, X\rangle J\,\mathrm{d}\varphi(\beta) - 2\langle\sigma, Y\rangle J\,\mathrm{d}\varphi(\alpha) \quad \text{by (3.6)}\\ &= -2J\,\mathrm{d}\varphi(\langle\sigma, X\rangle Y + \langle\sigma, Y\rangle X).\end{aligned}$$

□

Proposition 3.7. *Let α be any φ-horizontal vector field on S^3, and let $w = \mathrm{d}\varphi(\alpha)$. Then*

$$\mathcal{J}_\varphi(w) = \mathrm{d}\varphi(\mathcal{J}_\sigma^{\mathrm{v}}(\alpha)) - 2w + 4J\,\mathrm{d}\varphi(\nabla_\sigma\alpha).$$

Proof. It follows from (3.4) and (2.5) that

$$\mathcal{J}_\varphi(w) = \mathrm{d}\varphi(\mathcal{J}_\sigma^{\mathrm{v}}(\alpha)) - 2\sum_j \nabla \mathrm{d}\varphi(E_j, \nabla_{E_j}\alpha).$$

By Lemma 3.6 (applying the summation convention):

$$\begin{aligned}-2\nabla \mathrm{d}\varphi(E_j, \nabla_{E_j}\alpha) &= 4J\,\mathrm{d}\varphi(\langle E_j, \sigma\rangle\nabla_{E_j}\alpha + \langle\nabla_{E_j}\alpha, \sigma\rangle E_j)\\ &= 4J\,\mathrm{d}\varphi(\nabla_\sigma\alpha - \langle\alpha, \nabla_{E_j}\sigma\rangle E_j) = 4J\,\mathrm{d}\varphi(\mathrm{i}\alpha + \nabla_\sigma\alpha) \quad \text{by (3.2)}\\ &= -2\mathrm{d}\varphi(\alpha) + 4J\,\mathrm{d}\varphi(\nabla_\sigma\alpha) \quad \text{by (3.5)}.\end{aligned}$$

□

Proposition 3.8.
1. $\mathcal{F}_\sigma$ *and* $\mathcal{G}_\sigma$ *are complex subspaces of* $\mathcal{V}_\sigma$.
2. $\mathcal{F}_\sigma \subset \mathcal{Z}_\sigma \oplus \mathcal{P}_\sigma$.
3. $\mathcal{G}_\sigma \subset \mathcal{P}_\sigma$.

Note. Since the φ-horizontal distribution is a complex vector bundle, $\mathcal{V}_\sigma$ is a complex vector space.

Proof.

1. It was shown in [18] that $\mathrm{i}\mathcal{G}_\sigma = \mathcal{G}_\sigma$. On the other hand, $\mathfrak{C}$ may be viewed as the Lie algebra of holomorphic vector fields on the Riemann sphere, and is therefore a complex Lie algebra. It follows from (3.5) that $\mathrm{i}\tilde{\mathfrak{C}} = \tilde{\mathfrak{C}}$. Furthermore, $\mathrm{i}\sigma_2 = -\sigma_3$, so the subspace $\mathbb{R}\sigma_2 \oplus \mathbb{R}\sigma_3$ is also complex. It therefore follows from (3.3) that $\mathrm{i}\mathcal{F}_\sigma = \mathcal{F}_\sigma$.
2. If α is a φ-basic vector field on S^3 (i.e. a φ-horizontal field which projects to S^2), then $[\sigma, \alpha]$ is φ-adapted to the zero vector field on S^2, and hence φ-vertical. Therefore
$$\mathrm{d}\varphi(\nabla_\sigma \alpha) = \mathrm{d}\varphi(\nabla_\alpha \sigma) = \mathrm{d}\varphi(\mathrm{i}\alpha).$$
If in addition $\mathrm{d}\varphi(\alpha)$ is a Jacobi field for φ then Proposition 3.7 reads:
$$\mathrm{d}\varphi(\mathcal{J}_\sigma^{\mathrm{v}}(\alpha)) = 2\mathrm{d}\varphi(\alpha) - 4J\,\mathrm{d}\varphi(\mathrm{i}\alpha) = 4\mathrm{d}\varphi(\alpha) \quad \text{by (3.2)},$$
which implies $\mathcal{J}_\sigma^{\mathrm{v}}(\alpha) = 4\alpha$. In particular, this shows that $\tilde{\mathfrak{C}} \subset \mathcal{P}_\sigma$. On the other hand, if α is any left-invariant φ-horizontal vector field then $\mathcal{J}_\sigma^{\mathrm{v}}(\alpha) = 0$, since any such vector field is Hopf (see also Proposition 2.2, with f_2 and f_3 constant).
3. It was shown in [18] that every element of $\mathcal{G}_\sigma$ is an eigenvector of $\mathcal{J}_\sigma^{\mathrm{v}}$ with eigenvalue 1. (This can also be seen from Proposition 3.7, since the variation field $w = \mathrm{d}\varphi(\alpha)$ is a -1 eigenvector for $\mathcal{J}_\varphi$, see [19], and $\nabla_\sigma \alpha$ is φ-vertical. In fact, since $\mathcal{G}_\sigma$ is four-dimensional, it follows from Theorem 2.3 that $\mathcal{G}_\sigma$ is the entire eigenspace with eigenvalue 1).

□

A variation σ_t of σ through unit vector fields produces a variation $\varphi_t = \eta \circ \sigma_t$ of the Hopf map, by Proposition 3.1. However, as noted in (A) and (B) above, the most natural variations of φ with variation fields in $\mathcal{N}_\varphi$ or $\mathcal{Z}_\varphi$ are of the form $\varphi \circ \psi_t$, where $\{\psi_t\}$ is the flow of a (φ-horizontal) vector field on S^3. In order to play off Proposition 3.8 against Proposition 3.4 we need to relate the variation fields of these two types of variation (see Proposition 3.11).

Lemma 3.9. *If α is any φ-horizontal vector field on S^3 then* $\mathrm{d}\varphi(\alpha) = 2\eta(\mathrm{i}\alpha)$.

Proof. It follows from Lemma 3.2(2), with $Y = \sigma$, and Proposition 3.1 that
$$\mathrm{d}\varphi(\alpha) = \eta(\nabla_\alpha \sigma) + \tfrac{1}{2}\mathrm{Ad}(g)[\mu \circ \alpha, P_1].$$
Write $\alpha = f_2\sigma_2 + f_3\sigma_3$, so that $\mu \circ \alpha = f_2 P_2 + f_3 P_3$. The commutation relations (2.1) yield
$$[\mu \circ \alpha, P_1] = [f_2 P_2 + f_3 P_3, P_1] = 2 f_3 P_2 - 2 f_2 P_3 = 2\mu(\mathrm{i}\alpha).$$
Using (3.2), it follows that
$$\mathrm{d}\varphi(\alpha) = \eta(\mathrm{i}\alpha) + \mathrm{Ad}(g)\mu(\mathrm{i}\alpha) = 2\eta(\mathrm{i}\alpha). \qquad \square$$

Lemma 3.10. *If K is the connection map for any linear connection on a Lie group G, then*
$$\mathrm{d}\eta(V) = \eta(KV) \quad \text{for all vertical } V \in T(TG).$$

Proof. Let $g \in G$, and $X, Y \in T_g G$. Suppose V is the 'vertical lift' of Y at X:

$$V = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (X + tY).$$

Then $KV = Y$, and therefore $\eta(KV) = Y \cdot g^{-1}$. On the other hand,

$$\mathrm{d}\eta(V) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} ((X + tY) \cdot g^{-1}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (X \cdot g^{-1} + tY \cdot g^{-1}) = Y \cdot g^{-1}. \qquad \square$$

Proposition 3.11. *Let σ_t be a variation of the canonical Hopf vector field on S^3 through unit vector fields, with variation field α. Define two variations of the Hopf map φ as follows*:

$$\varphi_t = \eta \circ \sigma_t \quad \textit{and} \quad \Phi_t = \varphi \circ \psi_t,$$

where $\{\psi_t : t \in \mathbb{R}\}$ is the flow of the vector field $X = -\mathrm{i}\alpha$ on S^3. If w (resp. W) is the variation field of φ_t (resp. Φ_t) then $W = 2w$.

Proof. Let V denote the vertical lift of α into the tangent bundle of TS^3. Thus, for each $x \in S^3$, $V(\sigma(x))$ is the element of $T_{\sigma(x)}TS^3$ tangent to the curve $\sigma_t(x)$ in $T_x S^3$ at $t = 0$. Furthermore, $\alpha = KV$, by a characteristic property of connection maps. By Lemma 3.10, the variation field for φ_t is

$$w = \mathrm{d}\eta(V) = \eta(KV) = \eta(\alpha).$$

On the other hand, by Lemma 3.9:

$$W = \mathrm{d}\varphi(X) = -\mathrm{d}\varphi(\mathrm{i}\alpha) = 2\eta(\alpha). \qquad \square$$

To show that $\mathcal{N}_\sigma$ is trivial, we argue by contradiction. Suppose that σ_t is a variation of σ through unit vector fields, with variation field α a non-trivial negative eigenvector of $\mathcal{J}_\sigma^{\mathrm{v}}$. It follows from parts (2) and (3) of Proposition 3.8 that α is L^2-orthogonal to $\mathcal{F}_\sigma \oplus \mathcal{G}_\sigma$. Hence, by part (1) of Proposition 3.8, $X = -\mathrm{i}\alpha$ is also L^2-orthogonal to $\mathcal{F}_\sigma \oplus \mathcal{G}_\sigma$. Therefore $\mathrm{d}\varphi(X)$ is L^2-orthogonal to $\mathcal{Z}_\varphi \oplus \mathcal{N}_\varphi$, since elements of $\mathcal{F}_\sigma$ and $\mathcal{G}_\sigma$ are φ-horizontal, and φ is horizontally homothetic. Now $\mathrm{d}\varphi(X)$ is the variation field W for the variation $\Phi_t = \varphi \circ \psi_t$ of φ, where $\{\psi_t\}$ is the flow of X. Since $W \in \mathcal{P}_\varphi$, it follows from Proposition 3.11 that $w \in \mathcal{P}_\varphi$ also, where w is the variation field for $\varphi_t = \eta \circ \sigma_t$. Therefore, by the second variation formula for harmonic maps, φ_t is E-increasing for small $|t|$. However σ_t is E^{v}-decreasing in a neighbourhood of $t = 0$, and it therefore follows from Proposition 3.4 that φ_t is E-decreasing.

Remark 3.12. *The energy formula* (*Proposition* 3.4) *may be used to compute the spectrum of the Jacobi operator $\mathcal{J}_\varphi$ of the Hopf map, thereby correcting errors of* [15]. *Define $E^1(\sigma_t) = E(\eta \circ \sigma_t)$ and $E^2(\sigma_t) = E(\mu \circ \sigma_t)$, and let $\mathcal{J}_\sigma^1$ and $\mathcal{J}_\sigma^2$ denote the corresponding Jacobi operators at the Hopf vector field σ; note from Proposition* 3.1 *that $\mathcal{J}_\sigma^1$ is conjugate to $\mathcal{J}_\varphi$. It follows from Proposition* 3.4 *that*

$$\mathcal{J}_\sigma^1 = 2\mathcal{J}_\sigma^{\mathrm{v}} - \mathcal{J}_\sigma^2.$$

Write $\alpha \in \mathcal{V}_\sigma$ *as* $\alpha = f_2\sigma_2 + f_3\sigma_3$, *and define* $\psi : \alpha \mapsto f = f_2 + \mathrm{i} f_3$, *as in Section* 2. *Then f is a variation field for the constant map* $\mu \circ \sigma$, *whose Jacobi operator is the Laplace–Beltrami operator*

$$\psi \circ \mathcal{J}^2_\sigma \circ \psi^{-1}(f) = \Delta f.$$

Therefore it follows from Proposition 2.2 *that*

$$\psi \circ \mathcal{J}^1_\sigma \circ \psi^{-1}(f) = 2(\Delta f - 2\mathrm{i}\,\nabla_\sigma f) - \Delta f = \Delta f - 4\mathrm{i}\,\nabla_\sigma f.$$

By arguing as in the proof of Theorem 2.3, *it follows that the eigenvalues of* $\mathcal{J}_\varphi$ *are* $n(n+2) + 4k$, *where n is a non-negative integer and* $k = -n, -n+2, \dots, n-2, n$, *with multiplicity* $2(n+1)$. *In particular, the only negative eigenvalue is* $n = 1$, $k = -1$; *thus* $\mathcal{N}_\varphi$ *is four-dimensional. Furthermore, the only possibilities for eigenvalue zero are* $n = 0$, $k = 0$ *and* $n = 2$, $k = -2$. *The multiplicity of the former is* 2, *and that of the latter is* $2(2+1) = 6$; *thus* $\mathcal{Z}_\varphi$ *is indeed eight-dimensional*.

4. Proof of the Main theorem

First, recall that a vector field X on a Riemannian manifold (M, g) is said to define a *conformal foliation*, or *shear-free congruence*, if

$$L_X g(A, B) = \lambda g(A, B) \tag{4.1}$$

for all vector fields A, B pointwise orthogonal to X, where $\lambda : M \to \mathbb{R}$ is a smooth function. (It follows from (2.10) that $(n-1)\lambda = 2\,\mathrm{div}\,\sigma$, where n is the dimension of M.) If in addition the integral curves of X are (possibly reparametrized) geodesics, then X defines a *conformal geodesic foliation*, or *shear-free geodesic congruence*. Such vector fields appear naturally in the following non-linear version of Proposition 2.5.

Proposition 4.1 (Non-linear inequality). *Let* σ *be a unit vector field on an n-dimensional Riemannian manifold. Then*

$$(n-1)|L_\sigma g|^2 \geq 4(\mathrm{div}\,\sigma)^2$$

with equality if and only if σ *is a shear-free geodesic congruence*.

Proof. The argument used to derive this inequality is almost identical to that of Proposition 2.5; however, in place of identity (2.12), Eq. (2.9) may be used to derive

$$L_\sigma g(\sigma, \sigma) = 2\langle \nabla_\sigma \sigma, \sigma\rangle = \sigma \cdot |\sigma|^2 = 0.$$

(Note that it is not necessary to assume that the integral curves of σ are geodesics.) Thus Eq. (2.13) holds with $X = \sigma$.

Since σ has constant length, it follows from (2.9) that

$$L_\sigma g(\sigma, \alpha_i) = \langle \nabla_\sigma \sigma, \alpha_i\rangle,$$

so the vanishing of these Lie derivatives implies $\nabla_\sigma \sigma = 0$. On the other hand, the vanishing of the remaining Lie derivatives in (2.13) implies

$$L_\sigma g(\alpha, \beta) = \lambda\, g(\alpha, \beta)$$

for all α, β orthogonal to σ, where $\lambda : M \to \mathbb{R}$ is a smooth function. Thus, equality occurs only when σ defines a conformal geodesic foliation. □

Our Main theorem can now be proved by using the Bochner–Yano integral formula (2.7) and the following consequence of [2,3].

Rigidity theorem. *A unit vector field σ is a shear-free geodesic congruence on S^3 if and only if σ is a Hopf vector field.*

Note that the "if" part is obvious. We shall give a direct proof of this theorem later for completeness. Let us first prove our Main theorem using this result.

Main theorem. *The absolute minimum of E^{v} over all unit vector fields on S^3 is $2\pi^2$, which is achieved at, and only at, the Hopf vector fields.*

Proof. By the Bochner–Yano integral formula (2.7), we find on S^3 that

$$E^{\mathrm{v}}(\sigma) = \frac{1}{4}\int_{S^3} (|L_\sigma g|^2 - 2(\operatorname{div} \sigma)^2)\, \mathrm{d}x + 2\pi^2.$$

We have used $\mathrm{Ric}(\sigma, \sigma) = 2$ and the fact that the volume of S^3 is $2\pi^2$. By Proposition 4.1 with $n = 3$, we immediately have $E^{\mathrm{v}}(\sigma) \geq 2\pi^2$ and that the minimum is achieved if and only if σ is a shear-free geodesic congruence. Then, the theorem follows immediately from the Rigidity theorem. □

We conclude this paper by proving the "only if" part of the Rigidity theorem as promised. We first prove the following lemma.

Lemma 4.2. *Suppose that σ is a shear-free congruence on a Riemannian 3-manifold. Let α, β and σ form a local orthonormal frame, and define $Z = \alpha + \mathrm{i}\beta$. Then*

$$\nabla_Z \sigma = \Phi\, Z, \tag{4.2}$$

where $\Phi : M \to \mathbb{C}$ is a smooth complex function.

Proof. By definition (4.1) of shear-free congruences, we obtain

$$\langle \alpha, \nabla_\alpha \sigma \rangle = \langle \beta, \nabla_\beta \sigma \rangle = \tfrac{1}{2}\lambda = \mu \quad \text{(say)}$$

and

$$\langle \alpha, \nabla_\beta \sigma \rangle + \langle \beta, \nabla_\alpha \sigma \rangle = 0.$$

By defining $\nu = \langle \alpha, \nabla_\beta \sigma \rangle$ we find

$$\nabla_\alpha \sigma = \mu\alpha - \nu\beta, \qquad \nabla_\beta \sigma = \mu\beta + \nu\alpha.$$

Then, we obtain the desired formula by letting $\Phi = \mu + \mathrm{i}\nu$. □

A couple of differential equations can be derived for Φ on a manifold of constant curvature.

Proposition 4.3. *Let σ be a shear-free geodesic congruence on a 3-manifold of constant curvature c. Then*

$$\sigma \cdot \Phi = -c - \Phi^2, \tag{4.3}$$

$$\bar{Z} \cdot \Phi = 0, \tag{4.4}$$

where Φ and Z are defined in Lemma 4.2.

Proof. From (4.2) we find

$$\nabla_\sigma(\nabla_Z \sigma) = (\sigma \cdot \Phi)Z + \Phi\, \nabla_\sigma Z. \tag{4.5}$$

On the other hand, we have

$$\begin{aligned} \nabla_\sigma(\nabla_Z \sigma) &= R(\sigma, Z)\sigma + \nabla_Z(\nabla_\sigma \sigma) + \nabla_{\nabla_\sigma Z}\sigma - \nabla_{\nabla_Z \sigma}\sigma \\ &= c(\langle Z, \sigma\rangle \sigma - Z) + \nabla_Z(\nabla_\sigma \sigma) + \nabla_{\nabla_\sigma Z}\sigma - \nabla_{\nabla_Z \sigma}\sigma. \end{aligned} \tag{4.6}$$

Note that for any vector X

$$X = \tfrac{1}{2}(\langle \bar{Z}, X\rangle Z + \langle Z, X\rangle \bar{Z}) + \langle \sigma, X\rangle \sigma.$$

Using this formula, $\nabla_\sigma \sigma = 0$ and Eq. (4.2), we find from (4.6) that

$$\nabla_\sigma(\nabla_Z \sigma) = \Phi \nabla_\sigma Z - (c + \Phi^2)Z.$$

By comparing this with (4.5), we obtain (4.3).

Next we have

$$\begin{aligned} (\nabla_Z \nabla_{\bar{Z}} - \nabla_{\bar{Z}} \nabla_Z)\sigma &= \nabla_Z(\bar{\Phi}\bar{Z}) - \nabla_{\bar{Z}}(\Phi Z) \\ &= (Z \cdot \bar{\Phi})\bar{Z} + \bar{\Phi}\nabla_Z \bar{Z} - (\bar{Z} \cdot \Phi)Z - \Phi\, \nabla_{\bar{Z}} Z. \end{aligned} \tag{4.7}$$

On the other hand

$$(\nabla_Z \nabla_{\bar{Z}} - \nabla_{\bar{Z}} \nabla_Z)\sigma = R(Z, \bar{Z})\sigma + \nabla_{\nabla_Z \bar{Z}}\sigma - \nabla_{\nabla_{\bar{Z}} Z}\sigma = \bar{\Phi}\nabla_Z \bar{Z} - \Phi \nabla_{\bar{Z}} Z.$$

By comparing this formula with (4.7) we find

$$(Z \cdot \bar{\Phi})\bar{Z} - (\bar{Z} \cdot \Phi)Z = 0.$$

By taking the inner product with $\frac{1}{2}\bar{Z}$, we obtain (4.4). □

This proposition allows us to compute the Laplacian of Φ, as shown in the following lemma.

Lemma 4.4. *Suppose σ is a shear-free geodesic congruence on a* 3*-manifold of constant curvature. Then the function Φ is harmonic*.

Proof. Note first that

$$-\Delta\Phi = \nabla_\sigma \nabla_\sigma \Phi + \nabla_Z \nabla_{\bar{Z}} \Phi - \nabla_{\nabla_Z \bar{Z}} \Phi.$$

Then, using Proposition 4.3 and

$$\begin{aligned}\nabla_Z \bar{Z} &= \langle \sigma, \nabla_Z \bar{Z} \rangle \sigma + \tfrac{1}{2}(\langle \bar{Z}, \nabla_Z \bar{Z} \rangle Z + \langle Z, \nabla_Z \bar{Z} \rangle \bar{Z}) \\ &= -\langle \nabla_Z \sigma, \bar{Z} \rangle \sigma + \tfrac{1}{2}\langle Z, \nabla_Z \bar{Z} \rangle \bar{Z} = -2\Phi\sigma + \tfrac{1}{2}\langle Z, \nabla_Z \bar{Z} \rangle \bar{Z},\end{aligned}$$

we find $\Delta\Phi = 0$. □

The Rigidity theorem follows from this lemma.

Proof of the rigidity theorem. Since S^3 is compact and Φ is harmonic, Φ is constant. Then, from (4.3) we find $\Phi = \pm \mathrm{i}$ on S^3 with $c = 1$. Thus

$$\nabla_Z \sigma = \pm \mathrm{i} Z, \tag{4.8}$$

which implies that σ is a Hopf vector field. This fact can be seen as follows. Eq. (4.8) can be written as

$$\nabla_\alpha \sigma = \mp \beta, \qquad \nabla_\beta \sigma = \pm \alpha.$$

These equations, together with $\nabla_\sigma \sigma = 0$ imply

$$L_\sigma g(X, Y) = \langle X, \nabla_Y \sigma \rangle + \langle Y, \nabla_X \sigma \rangle = 0$$

for any vector fields X and Y. Thus, σ is a Killing vector field of unit length, hence a Hopf vector field. □

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